The Optimal Spacing of Interfering Wells: An Analytic Solution

by John Briscoe[a]

ABSTRACT

An exact expression is derived for the optimal spacing between interfering wells in a rectangular well field in an ideal confined aquifer. A simple, practical method for determining the optimal spacing is presented. The optimal spacing is shown to be substantially different from the spacing determined by use of the Theis formulation. The economic savings resulting from use of the revised approach are evaluated and found to be considerable, especially when the number of wells is large and the transmissivity of the aquifer low.

1. INTRODUCTION

It has long been recognized that the optimal spacing of pumped wells in a well field involves a trade-off between pumping costs (which increase as well spacing is reduced and interference increases) and the costs of connecting pipelines (which decrease as well spacing is reduced).

The most comprehensive treatment of the subject of optimal well spacing was presented by Hantush (1961 and 1964). Hantush derived expressions for four different cases, namely a group of three wells forming an equilateral triangle, and groups of two, three, and four wells equally-spaced along a straight line. In all cases, Hantush assumed that the wells each were pumped at the same rate, and that the annualized cost per unit length of the connecting pipeline was constant.

In 1963 Theis undertook a similar analysis, examining only the two-well case, although he made a special note that "if more than two wells are to be pumped, the analysis will have to be modified to take (this factor) into account" (Theis, 1963).

Since these initial investigations, the problem of an analytic solution to the optimal well spacing problem has received little further attention. Many current ground-water textbooks (e.g. Todd, 1980; and Freeze and Cherry, 1979) ignore the well-spacing problem entirely. Others (e.g. Heath, 1980) present the Theis formulation as a general solution to the problem of optimal well spacing, ignoring the caveats put on the use of the formulation by Theis himself.

The purpose of the present analysis is to assess the appropriateness of the Theis formulation for the optimal spacing of a series of wells in a rectangular well field in a confined homogeneous aquifer with boundaries at infinity. It is assumed that all wells are pumped at the same rate, and that the cost of each section of connecting pipeline depends on the flow to be carried in that section.

2. NOTATION

The symbols in this paper are defined where they first appear. They are assembled alphabetically for convenience in Appendix I.

3. WELL FIELD COSTS

Following Hantush (1961), the yearly cost of operating a well field may be expressed as:

Total annual cost of operation		Annualized cost of connecting pipelines		Annual cost of pumping against additional head caused by interference
	=		+	

or

$$C_T = C_P + C_I$$

where

C_T is the total yearly cost of connecting pipelines and of operation as affected by well interference;

C_P is the annualized cost for maintenance, depreciation, and original cost of connecting pipelines;

C_I is the annual cost of pumping against additional head caused by interference; and

$$C_T = C_P + cqt_o \sum_{i=1}^{M} \sum_{j=1}^{N} s_{ij} \qquad \text{(A)}$$

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Received May 1984, accepted May 1984.
Discussion open until March 1, 1985.

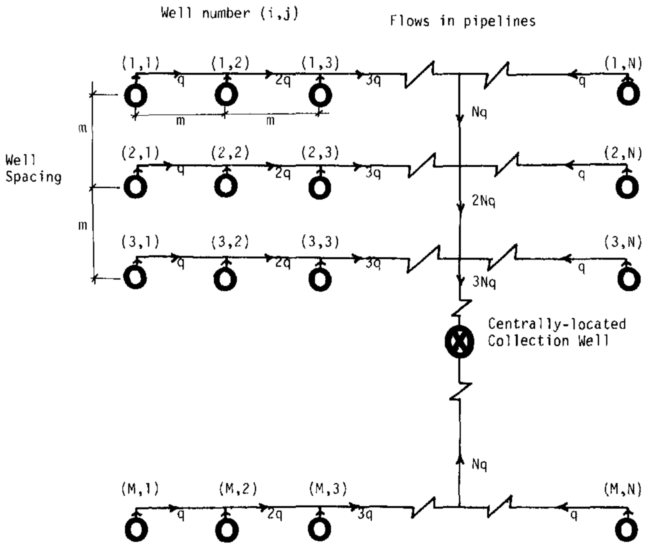

Fig. 1. Layout of wells in a rectangular well field.

where

- C_P is the total annualized cost of the piping system;
- c is the unit cost of raising a unit volume of water a unit height;
- q is the rate at which each well in the field is pumped;
- N is the number of "columns" of wells;
- M is the number of "rows" of wells;
- s_{ij} is the total drawdown in well (i,j) due to pumping of all the other wells; and
- t_o is the period of continuous pumping.

The optimal value of the well spacing, m*, is determined by solving the equation

$$\frac{\partial C_P}{\partial m} + cqt_o \frac{\partial}{\partial m} \sum_j \sum_i s_{ij} = 0, \qquad \text{(B)}$$

and checking that the appropriate second-order conditions are met.

4. DRAWDOWN EQUATIONS

The rectangular well field considered in this analysis is illustrated on Figure 1.

The aquifer is assumed to be ideal (homogeneous, isotropic, with boundaries at infinity), and confined, with the wells fully penetrating. The first task is to develop an expression, s_{ij}, representing the drawdown in the (i,j)th well due to pumping in all of the other wells. To do this, first consider the general case presented in Figure 2.

Using the Cooper-Jacob expression (which is valid for small r and large t) for the drawdown in the observation well,

$$s = \frac{2.3q_a}{4\pi T} \log \frac{2.25Tt}{(r_a)^2 S}$$

where

- s is the drawdown in observation well due to pumping well "a";
- q_a is the rate at which water is pumped from a well a distance r_a from the observation well;
- T is the transmissivity of the aquifer;
- t is the time since pumping began;
- S is the storage coefficient in the aquifer; and
- r_a is the distance between the observation well and the (a)th pumping well.

Applying the principle of superposition, the total drawdown in the observation well due to pumping of wells "a" and "b" is:

$$s = \frac{2.3q_a}{4\pi T} \log \frac{2.25Tt}{r_a^2 S} + \frac{2.3q_b}{4\pi T} \log \frac{2.25Tt}{r_b^2 S}$$

for $q_a = q_b = q$

$$s = \frac{2.3q}{4\pi T} (\log \frac{2.25Tt}{r_a^2 S} + \log \frac{2.25Tt}{r_b^2 S})$$

$$= \frac{2.3q}{4\pi T} \times 2[\log \frac{2.25Tt}{(r_a^2 \cdot r_b^2)^{1/2} S}]$$

i.e., the drawdown in the observation well is precisely the same as the drawdown which would result from a single well being pumped at 2q and located a distance $(r_a \cdot r_b)^{1/2}$ from the observation well.

Similarly, it can be shown easily that the drawdown in an observation well due to pumping in N other wells located $r_1, r_2, \ldots r_N$ from the

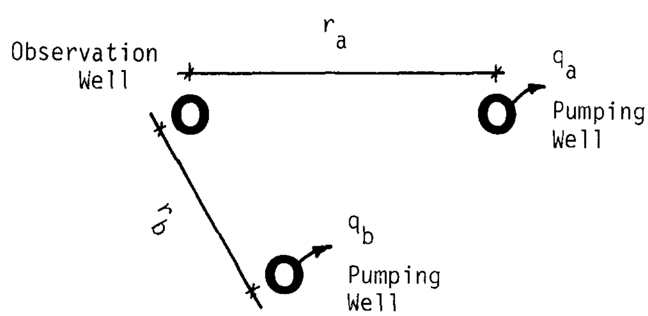

Fig. 2. The effect of pumping on an observation well.

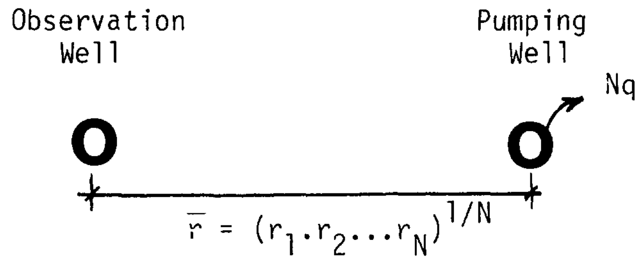

Fig. 3. Equivalent well system.

observation well is equivalent to the drawdown in the system shown in Figure 3.

Returning to the case of the rectangular well field in Figure 1, the drawdown in well (i, j) due to pumping in the other (MN-1) wells is equivalent to the drawdown from a single well pumping q(MN-1) located a distance $\bar{m}_{ij}$ from well (i,j) where $\bar{m}_{ij}$ is the geometric mean of the distances of all other wells from well (i,j).

The distance of well (k,l) from well (i,j) is

$$\sqrt{(i-k)^2 m^2 + (j-l)^2 m^2} = m\sqrt{(i-k)^2 + (j-l)^2}$$

whence

$$\bar{m}_{ij} = \{ \prod_{\forall k} \prod_{\forall l} m[(i-k)^2 + (j-l)^2]^{1/2} \}^{1/MN-1} \quad \text{except } (k,l) = (i,j)$$

i.e.,

$$\bar{m}_{ij} = m\{ \prod_{k} \prod_{l} [(i-k)^2 + (j-l)^2] \}^{1/2(MN-1)} \quad \text{except } (k,l) = (i,j) \qquad \text{(C)}$$

Recall, from equation (A), that in determining the total additional cost of pumping due to interference, it is necessary to evaluate the expression $\sum_{i=1}^{M} \sum_{j=1}^{N} s_{ij}$, that is, the total additional drawdown in all of the wells. Using (C),

$$\Sigma\Sigma s_{ij} = \Sigma\Sigma [\frac{2.3q(MN-1)}{4\pi T} \log \frac{2.25Tt}{\bar{m}_{ij}^2 S}]$$

$$= \frac{2.3q(MN-1)}{4\pi T} \Sigma\Sigma \log \frac{2.25Tt}{\bar{m}_{ij}^2 S}$$

$$= \frac{2.3q(MN-1)}{4\pi T} MN \log \frac{2.25Tt}{S(\prod_{i=1}^{M} \prod_{j=1}^{N} \bar{m}_{ij})^{1/MN}}$$

That is, the total drawdown in the system ($\Sigma\Sigma s_{ij}$) is equivalent to the drawdown induced by a single well pumping [q(MN-1) · MN] at a distance $\bar{m}_{ave}$ from the well, where

$$\bar{m}_{ave} = (\prod_{i=1}^{M} \prod_{j=1}^{N} \bar{m}_{ij})^{1/MN}$$

i.e.,

$$\bar{m}_{ave} = m\{\prod_i \prod_j \{ \prod_k \prod_l [(i-k)^2+(j-l)^2] \}^{1/2(MN-1)} \}^{1/MN} \quad \text{except } (k,l)=(i,j)$$

$$= m\{\prod_i \prod_j \prod_k \prod_l [(i-k)^2+(j-l)^2] \}^{1/2MN(MN-1)} \quad \text{except } (k,l)=(i,j)$$

$$= m \cdot f(M,N)$$

and

$$\Sigma\Sigma s_{ij} = \frac{2.3q(MN-1)MN}{4\pi T} \log \{ \frac{2.25Tt}{Sm^2 f(M,N)^2} \}$$

whence

$$\frac{\partial}{\partial m} \Sigma\Sigma s_{ij} = -\frac{q(MN-1)MN}{2\pi Tm} \qquad \text{(D)}$$

5. PIPELINE COST FUNCTION

At present prices, the optimal velocity of water in a transmission pipeline is about 1.2 m/sec (Olson, 1976), whence

$$D = 0.0035 q^{1/2}$$

where D is the pipeline diameter in m; and q is the flow in m^3/day.

The capital cost of an installed pipeline (y, in \$/m) can be related to the diameter of the pipeline (in m) through the equation

$$y = 392 D^{1.3} \qquad \text{(Deb, 1978)},$$

whence

$$y = 0.25 q^{0.65} \qquad (\$/m).$$

Assuming an effective life of 30 years for the pipeline and an interest rate of 12% per annum,

$$\text{Capital Recovery Factor} = \frac{0.12(1.12)^{30}}{1.12^{30} - 1} = 0.124$$

whence

$$y_a = 0.03 q^{0.65} \qquad (\$/yr/m)$$

or, in more general form,

$$y_a = bq^{0.65}$$

where b is the cost of a unit length of pipe carrying a unit flow (= 0.03 in 1978 prices); and y_a is the annualized cost of a unit length of the pipeline.

The flows in the pipe network are shown in Figure 1. Consider first the cost of the connecting (E-W) pipes for any row of wells on Figure 1. The cost of the pipes for one row is

$$bm\,[(q)^{0.65} + (2q)^{0.65} + \ldots + (2q)^{0.65} + (q)^{0.65}]$$

$$= bmq^{0.65}\,(2 \sum_{j=1}^{(N-1)/2} j^{0.65}) \text{ for odd N.}$$

Therefore, for odd N and odd M,

$$\text{Cost of E-W pipelines} = M\,mbq^{0.65}\,(2 \sum_{j=1}^{(N-1)/2} j^{0.65})$$

and

$$\text{Cost of N-S pipelines} = mb\,(Nq)^{0.65}\,(2 \sum_{j=1}^{(M-1)/2} j^{0.65})$$

Thus, the total cost of the pipeline is

(a) for M_{odd} and N_{odd}:

$$bq^{0.65}\,m\,[2M \sum_{j=1}^{(N-1)/2} j^{0.65} + 2N^{0.65} \sum_{j=1}^{(M-1)/2} j^{0.65}]$$

and (b) for M_{odd} and N_{even}:

$$bq^{0.65}\,m\,\{M\,[2 \sum_{j=1}^{(N-2)/2} j^{0.65} + (N/2)^{0.65}] + N^{0.65}\,2 \sum_{j=1}^{(M-1)/2} j^{0.65}\}$$

6. OPTIMAL WELL SPACING

Substituting from equation (D) and the differential of the pipeline cost expression into equation (B), (for M_{odd} and N_{odd}):

$$bq^{0.65}\,[2M \sum_{j=1}^{(N-1)/2} j^{0.65} + 2N^{0.65} \sum_{j=1}^{(M-1)/2} j^{0.65}] - cqt_o \frac{q(MN-1)MN}{2\pi Tm^*} = 0,$$

whence

$$m_A^* = \frac{cq^{1.35}t_o}{6.3Tb}\,[\frac{(MN-1)MN}{2M \sum_{j=1}^{(N-1)/2} j^{0.65} + 2N^{0.65} \sum_{j=1}^{(M-1)/2} j^{0.65}}] \quad \ldots\ldots\ldots \text{(E)}$$

7. COMPARISON WITH THEIS TWO-WELL FORMULATION

The Theis formula for the optimal spacing between wells is often used (see, for example, Heath, 1980) for multiple-well fields, despite the fact that Theis specifically limited the application of his formula to the case of two wells. An assessment will be made of the magnitude of the error involved in using the Theis two-well formulation for the optimal spacing between wells rather than expression (E).

For N=2, M=1, the Theis conditions, expression (E), with the appropriate expression for the cost of connecting pipelines, gives

$$m_T^* = \frac{2cq^{1.35}t_o}{6.3bT} \qquad \text{(F)}$$

(If the standard form for pipeline cost, viz, cost = k × m, rather than the form actually used, viz, cost = m × $bq^{0.65}$, then equation (F) reduces to:

$$m_T^* = \text{const.} \frac{q^{1.35}}{bT} \times \frac{bq^{0.65}}{k}$$

$$= \text{const. } q^2/kT$$

which is the Theis expression for optimal well spacing.)

From (E) and (F), for M_{odd} and N_{odd}

$$\frac{m_A^*}{m_T^*} = \frac{MN(MN-1)}{4(M \sum_{j=1}^{(N-1)/2} j^{0.65} + N^{0.65} \sum_{j=1}^{(M-1)/2} j^{0.65})} \qquad \text{(G)}$$

The expression (G) is evaluated for a variety of values of M and N and the results presented on Figure 4, which gives a simple method for calculating the optimal well spacing in a rectangular well field:

(1) m_T^* can be calculated using the standard Theis formula [see Theis (1963), Walton (1970), Heath (1980), or equation (F) above].

(2) m_A^*/m_T^* can be read off Figure 4 for any M and N;

(3) m_A^*, the actual optimal well spacing, is the product of the numbers derived in (1) and (2) above.

8. SENSITIVITY ANALYSIS

In his original analysis of the optimal spacing of two wells, Theis (1963) stated that "the graph of corresponding values of the total annual cost and the distance between two discharging wells is a curve that is relatively flat in the neighborhood of the minimum value; placing wells at distances that are somewhat more or less than that found (by the equation) would not appreciably increase the cost."

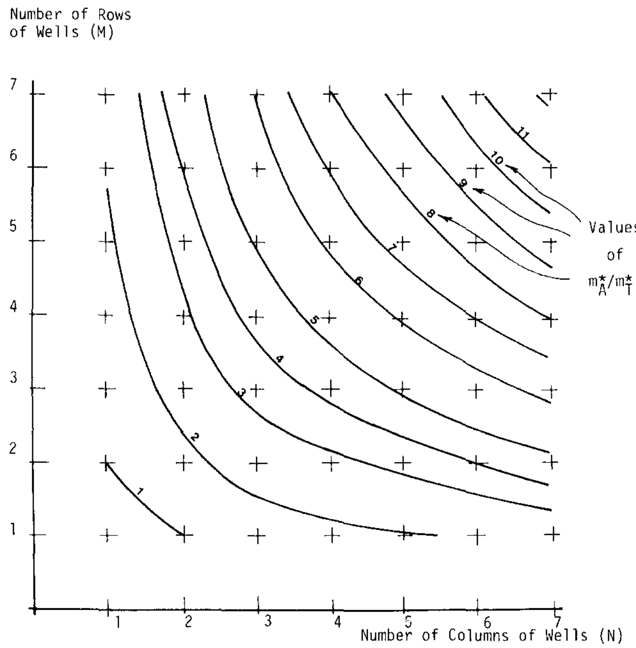

Fig. 4. Ratio of optimal well spacing to Theis well spacing for a rectangular well field.

In the present analysis it has been shown that, as the number of wells increases in a well field, the Theis formulation dramatically underestimates the optimal spacing between adjacent wells. The important question, however, is not so much whether m_T^* and m_A^* are different, but rather what the economic cost of using m_T^* instead of m_A^* may be. This sensitivity analysis can be undertaken only for specific aquifer conditions. Accordingly, the analysis was undertaken for aquifers with transmissivities typical of high-yielding aquifers in the coastal plain of the southeastern United States, viz,

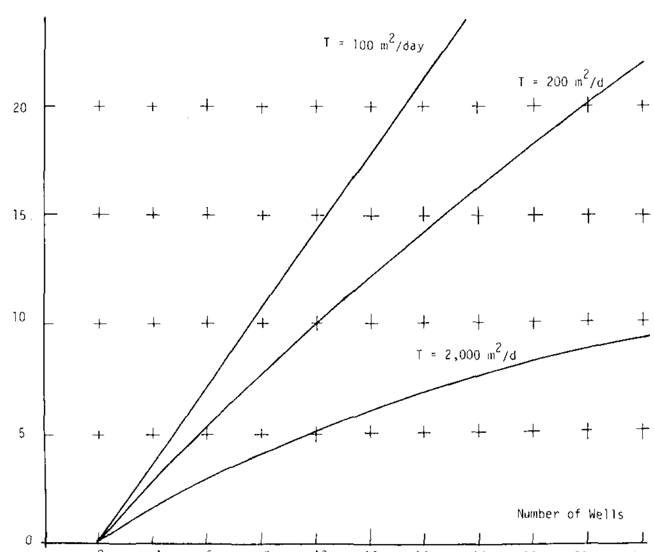

Fig. 5. Economic cost of using m_T^* instead of m_A^* in a linear well field.

$T = 100$ m²/day, 200 m²/day, and 2,000 m²/day. It was further assumed that: $S = 0.002$, $t_o = 365$ days, $q = 1{,}900$ m³/day, $b = 0.03$, and $c = 0.17 \times 10^{-3}$ \$/m³ (corresponding to \$0.03/kwh and a combined pump and motor efficiency of 50%).

For a variety of values of M and N, the values of m_A^* and m_T^* were calculated from equations (E) and (F) and the following expression for total annualized cost evaluated:

$$C_T(m,M,N) = bq^{0.65}\, m \left[2M \sum_{j=1}^{(N-1)/2} j^{0.65} + 2N \sum_{j=1}^{(M-1)/2} j^{0.65}\right] + 0.18 \frac{cq^2}{T} t_o\, MN(MN-1) \cdot \log \frac{2.25Tt}{Sm^2 \left\{\prod_{i}\prod_{j}\prod_{k}\prod_{l} \left[(i-k)^2+(j-l)^2\right]\right\}^{1/[2MN(MN-1)]}} \quad \text{except } (k,l)=(i,j)$$

The loss of economic efficiency (Thomas, 1971) due to the use of the incorrect spacing (m_T^*) rather than the correct spacing (m_A^*) is determined by calculating

$$\%\text{ loss of economic efficiency} = \frac{C(m_T^*, M, N) - C(m_A^*, M, N)}{C(m_A^*, M, N)}$$

Considering the case of $M = 1$ (a linear well field), the loss of economic efficiency is determined for different T and N and the results presented on Figure 5. (Since fixed costs were not included in the cost equations, the efficiency losses indicated on Figure 5 are thus somewhat inflated.) Figure 5 shows that, when the number of wells is small, the loss of economic efficiency from use of m_T^* insteady of m_A^* is small despite large differences between m_T^* and m_A^*, confirming Theis' observation in this range. As the number of wells increases, however, the loss of economic efficiency becomes substantial, especially for aquifers of low transmissivity, and it becomes essential to use the revised formulation for optimal well spacing (m_A^*) rather than the formulation developed by Theis for the two-well case (m_T^*).

Turning to the general case of a rectangular well field, it emerges that the loss of economic efficiency for a total of P wells in the rectangular case is similar to the loss of economic efficiency for P wells in the linear case. That is, as in the linear case, the losses increase as the total number of wells increases and as the transmissivity of the aquifer decreases. Thus, when there are several wells in a field and when the transmissivity of the aquifer is not extraordinarily high, substantial savings are obtained by using the revised optimal well spacing (m_A^*) rather than the spacing based on Theis' analysis of the two-well case (m_T^*).

9. CONCLUSION

An expression has been derived analytically for the optimal spacing between wells in a rectangular field in an ideal, confined aquifer. It has been shown that the optimal spacing is substantially different from the spacing determined by use of the Theis formulation and that the economic savings resulting from use of the revised approach are substantial, especially when the number of wells is large and the transmissivity of the aquifer low.

APPENDIX I. NOTATION

The following symbols are used in this paper:

b	The cost of a unit length of pipe carrying a unit flow.
c	Cost of raising a unit volume of water a unit length, consisting largely of power charges, but also properly including some additional charges on the equipment.
C_I	Annualized cost of pumping against additional head caused by interference.
C_P	Annualized cost for maintenance, depreciation, and original cost of connecting pipelines.
C_T	Total yearly cost of operation as affected by well interference and of connecting pipelines.
D	Diameter of pipeline.
f(M,N)	A function depending only on M and N.
i,j	Subscript for well in ith row, jth column.
m	Spacing between adjacent wells.
m^*	Optimal spacing between adjacent wells.
m_A^*	Actual optimal spacing.
m_T^*	Optimal spacing from Theis formulation.
$\bar{m}_{ij}$	Distance from (i,j)th well to "equivalent" pumping well.
$\bar{m}_{ave}$	Equivalent distance for computing total drawdown in well field.
M	Number of rows of wells.
N	Numbers of columns of wells.
q	Flow rate.
r	Distance from pumping to observation well.
s_{ij}	Drawdown in well (i,j) due to pumping in other wells.
S	Storage coefficient.
T	Transmissivity.
t_o	Period of continuous pumping.
y	Capital cost of pipeline.
y_a	Annualized cost of pipeline.

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